

# An exact derivation of the dissipation rate correlation exponent $\mu$ in fully-developed turbulence

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## Abstract

We derive for the Navier-Stokes equation an exact equation satisfied by the dissipation rate correlation function,  $\langle \epsilon(\vec{x} + \vec{r}, t + \tau) \epsilon(\vec{x}, t) \rangle$ . In the equal time limit, for the homogeneous, isotropic state of fully-developed turbulence, we show that the correlation function behaves as  $Ar^{-\mu_1} + Br^{-\mu_2}$  with  $\mu_1 = 2 - \zeta_6$  and  $\mu_2 = z_4'' - \zeta_4$  for  $r$  in the inertial range; the  $\zeta$ 's are exponents of velocity structure functions and  $z_4''$  is a dynamical exponent characterizing the 4th order dynamical structure function. This provides the first direct derivation of the exponents of the dissipation-rate correlation.

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The statistical properties of the energy dissipation rate  $\epsilon(\vec{x}, t)$  defined by

$$\epsilon(\vec{x}, t) = \frac{\nu}{2} \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 \quad (1)$$

where  $\partial_i = \partial/\partial x_i$  have played a crucial role in our understanding of fully-developed turbulence in incompressible fluids.[1, 2] In the original Kolmogorov theory  $\epsilon$  is replaced by  $\langle \epsilon \rangle$  and the spatial fluctuations are ignored; the effect of fluctuations of  $\epsilon$  pointed out by Landau have been explored in the context of the lognormal model and its multifractal generalizations.[3] The intermittent behavior of turbulent fluctuations is reflected in the power-law behavior of the correlations of  $\epsilon$ :

$$\langle \epsilon(\vec{x}) \epsilon(\vec{x} + \vec{r}) \rangle \sim (r/L)^{-\mu} \quad (2)$$

where  $L$  is a length scale characteristic of the large-scale flow and  $r = |\vec{r}|$  belongs to the inertial range. Simple dimensional analysis, noting that the dimension of  $\epsilon$  is  $V^3/L$ , yields the identification  $\mu = 2 - \zeta_6$ ; the exponents of the  $q^{th}$ -order (longitudinal) structure function,  $\zeta_q$  are defined by

$$S_q \equiv \langle [\delta \vec{u} \cdot \hat{r}]^q \rangle \sim (r/L)^{\zeta_q} \quad (3)$$

where  $\delta \vec{u} = \vec{u}(\vec{x} + \vec{r}, t) - \vec{u}(\vec{x}, t)$ . Within the original Kolmogorov theory  $\zeta_6 = 2$  and consequently  $\mu = 0$ . Thus the deviation of  $\mu$  from zero is a measure of the degree of intermittency and is an important quantity for understanding fully developed turbulence. This breakdown of simple Kolmogorov scaling has been studied experimentally and a review of the experiments[4] gives a “best” estimate for  $\mu$  of  $0.25 \pm 0.05$  which is consistent with the experimentally measured value of  $\zeta_6$ . In this Letter we provide a simple and direct derivation of the values of  $\mu$  for the Navier-Stokes equation by deriving the exact equation satisfied by the dissipation rate correlations. The equation for the dissipation rate contains two contributions, one a second spatial derivative of an appropriate sixth-order structure function and another a second temporal derivative of a fourth-order structure function; there are, in addition, pressure-dependent terms and no other velocity-dependent terms. The use of *dynamical* structure functions, i.e.,  $S_q$  in Eqn. (3) defined with  $\delta \vec{u} = \vec{u}(\vec{x} + \vec{r}, t + \tau) - \vec{u}(\vec{x}, t)$ , is key to our derivation. Both spatial and temporal derivatives of the dynamic structure functions occur naturally and

the equal time limit of the derivatives is related to the correlation of the energy dissipation rates. We illustrate the method first with an application to the 1d stochastic Burgers equation[5] which provides an extremely fruitful model for elucidating some of the conceptual and mathematical features of 3d turbulence. After providing the derivation in the Navier-Stokes case we comment on our results and discuss some experimental implications.

*Derivation for the stochastic Burgers equation:* We discuss first the Burgers equation (describing a one-dimensional, compressible fluid without pressure) driven by a stochastic force  $f(x, t)$ :

$$\partial u / \partial t + u \partial u / \partial x = \nu \partial^2 u / \partial x^2 + f \quad .$$

The stochastic force is a Gaussian white-noise with algebraic spatial correlations given in  $k$ -space by

$$\langle \hat{f}(k, t) \hat{f}(k', t') \rangle = D_0 |k|^\beta \delta_{k+k', 0} \delta(t - t') \quad (4)$$

with  $-1 \leq \beta < 0$ . For these values of  $\beta$ , intermittent behavior of the structure functions has been shown to occur.[6] We will first illustrate our approach by doing the simplest calculation. We employ the notation  $u = u(x, t)$ ,  $u' = u(x', t')$  where  $x = R + r/2$ ,  $x' = R - r/2$ ,  $t = T + \tau/2$ , and  $t' = T - \tau/2$ . We wish to find an equation for  $\langle \epsilon \epsilon' \rangle$  where  $\epsilon = \nu (\partial u / \partial x)^2$  is the dissipation rate. Multiplying the Burgers equation by  $u$  we have

$$\nu u \partial^2 u / \partial x^2 + f u = u \partial u / \partial t + u^2 \partial u / \partial x$$

and the corresponding equation for  $u'$ . The idea is simply to multiply the two equations to obtain

$$[\nu u \partial^2 u / \partial x^2 + f u] [\nu u' \partial^2 u' / \partial x'^2 + f' u'] = [u \partial u / \partial t + u^2 \partial u / \partial x] [u' \partial u' / \partial t' + u'^2 \partial u' / \partial x'] \quad (5)$$

and average over the homogeneous, steady state of Burgers turbulence; the identity  $\nu u \partial^2 u / \partial x^2 = \nu \partial^2 (u^2 / 2) / \partial x^2 - \epsilon$  is crucial. We obtain after some straightforward rearrangements of the terms on the right-hand side using  $\partial / \partial t = \partial / \partial \tau$  and  $\partial / \partial t' = -\partial / \partial \tau$  when acting on averages in the steady state,

$$\langle \epsilon \epsilon' \rangle - \langle \epsilon \rangle^2 \approx -\frac{1}{4} \frac{\partial^2}{\partial \tau^2} \langle u^2 u'^2 \rangle - \frac{1}{6} \frac{\partial^2}{\partial \tau \partial r} \langle (u + u') u^2 u'^2 \rangle - \frac{1}{9} \frac{\partial^2}{\partial r^2} \langle u^3 u'^3 \rangle \quad (6)$$

All the terms in the above equation are assumed to be evaluated in the limit  $\tau \rightarrow 0^+$ . The *only* reason for the approximate sign is that we have not displayed terms which depend explicitly on  $\nu$  or the noise and are negligible in the inertial range as  $\nu \rightarrow 0$ : for example, a term such as  $\nu^2 \partial_r^4 S_4$  is negligible since  $S_4$  is finite. In the equal-time limit the noise-velocity correlations that occur on the left-hand side of Eqn. (5) can be evaluated by using the Donsker-Novikov-Varadhan result[7]; of these the only term that does not vanish in the inertial range is the subtracted term  $\langle \epsilon \rangle^2$  in Eqn. (6).

We immediately observe that we obtain the second temporal derivative of a fourth-order structure function and the second spatial derivative of a sixth-order structure function which yield the two exponents referred to in the abstract. Of course, the equation is not manifestly form-invariant under Galilean transformations and we have to understand the role of the cross spatio-temporal derivative. We address this issue using a different version of the above equation.

A more elegant form can be obtained by multiplying Burgers equations for  $u$  and  $u'$  by  $\delta u = u - u' \equiv u(x, t) - u(x', t')$ , multiplying the two equations and averaging over the homogeneous turbulent state as before. We obtain

$$\langle \delta u (\nu \partial^2 u + f) \delta u (\nu \partial'^2 u' + f') \rangle = \langle \delta u \frac{Du}{Dt} \delta u \frac{Du'}{Dt'} \rangle \quad (7)$$

where  $D/Dt$  represents the convective derivative, i.e.,  $Du/Dt = \partial u/\partial t + u \partial u/\partial x$ , and we have used the shorthand notation  $\partial \equiv \partial/\partial x$  and  $\partial' \equiv \partial/\partial x'$ . Evaluating the purely viscous terms on the left-hand side of Eq. (7) yields

$$\nu^2 \langle (\delta u)^2 \partial^2 u \partial'^2 u' \rangle = -2 \langle \epsilon \epsilon' \rangle + \frac{\nu^2}{12} \partial_r^4 \langle (\delta u)^4 \rangle + \nu \partial_r^2 \langle (\epsilon + \epsilon') (\delta u)^2 \rangle$$

where the only non-vanishing term in the inertial range is  $-2 \langle \epsilon \epsilon' \rangle$  in the limit  $\nu \rightarrow 0$  since as before the rest are products of powers of  $\nu$  and finite correlation functions. The noise terms can again be evaluated in the  $\tau \rightarrow 0^+$  limit to yield  $2 \langle \epsilon \rangle^2 - (1/72)(\partial S_3/\partial r)^2$  apart from terms which vanish as  $\nu \rightarrow 0$ . The right-hand side of Eq. (7) contains four terms and their evaluation is facilitated by kinematic results which can be obtained in a straightforward way such as

$$\partial_\tau^2 \langle (\delta u)^4 \rangle = 12 \langle (\delta u)^2 \partial_t u \partial_{t'} u' \rangle \quad (8a)$$

$$\partial_r^2 \langle (\delta u)^6 \rangle = 30 \langle (\delta u)^4 \partial_x u \partial_{x'} u' \rangle \quad (8b)$$

$$\partial_r^2 \langle (u + u')^2 (\delta u)^4 \rangle = \langle [-2(\delta u)^4 + 12(u + u')^2 (\delta u)^2] \partial_x u \partial_{x'} u' \rangle \quad (8c)$$

These enable one to simplify the temporal and spatial second derivatives. The term involving cross (space and time) derivatives can be simplified by employing

$$\partial_\tau \partial_r \langle (\delta u)^4 u \rangle = \langle [12(\delta u)^2 u + 4(\delta u)^3] \partial_x u \partial_{t'} u' \rangle \quad (9a)$$

$$\partial_\tau \partial_r \langle (\delta u)^4 u' \rangle = \langle [12(\delta u)^2 u' - 4(\delta u)^3] \partial_{x'} u' \partial_t u \rangle \quad (9b)$$

$$\partial_\tau \partial_r \langle (\delta u)^5 \rangle = 20 \langle (\delta u)^3 \rangle \partial_x u \partial_{t'} u' = 20 \langle (\delta u)^3 \rangle \partial_{x'} u' \partial_t u. \quad (9c)$$

We obtain after a few manipulations

$$\langle \epsilon \epsilon' \rangle - \langle \epsilon \rangle^2 \approx -\frac{1}{24} [\partial_\tau^2 S_4 + \partial_\tau \partial_r S_{4,1} + \frac{1}{4} \partial_r^2 S_{4,2}] + \frac{1}{288} \partial_r^2 S_6 - \frac{1}{144} (\partial_r S_3)^2 \quad (10)$$

where we have used the notation  $S_{p,q} = \langle (u - u')^p (u + u')^q \rangle$ . Note that  $\partial_\tau^2 S_4 + \partial_\tau \partial_r S_{4,1} + \frac{1}{4} \partial_r^2 S_{4,2}$  is the convective second derivative of the fourth-order structure function,  $D^2 S_4 / D\tau^2$ . This term is manifestly Galilean-invariant as is evident if we recall that  $\partial_\tau \langle (u - u')^n \rangle + (1/2) \partial_r \langle (u + u')(u - u')^n \rangle$  is Galilean-invariant. The last term on the right-hand side which arises from the noise-velocity correlations leads to  $r^{2\zeta_3-2}$  which yields for the dissipation-rate correlation exponent precisely the scaling value of  $2 - \zeta_6$ ; the explicit value is given by  $2 - 2\zeta_3 = 2 + 2\beta$  where the last equality follows from the von-Karman-Howarth relation derived previously by us[8],  $\zeta_3 = -\beta$ . In the multifractal regime this term is, of course, subdominant. The crucial feature of Eq. (10) is that the non-scaling behavior of the dissipation rate correlations is determined by (1)  $\partial_r^2 S_6$  term which leads to  $\mu_1 = 2 - \zeta_6$  and (2)  $D^2 S_4 / D\tau^2$ . Let us define  $z_4''$  to be the dynamical scaling exponent describing the  $\tau \rightarrow 0$  limit of the Galilean-invariant, convective second derivative of  $S_4$ . This yields another possible inertial range behavior for  $\langle \epsilon \epsilon' \rangle$  with  $\mu_2 = z_4'' - \zeta_4$ . The need for a sequence of dynamical exponents is a consequence of the occurrence of temporal multiscaling in the dynamical structure functions as has been emphasized earlier[9]; thus different order temporal derivatives of  $S_p(r, \tau)$  can lead to different dynamical exponents. In dealing with dynamic structure functions it is important to recall that we have used the Eulerian description; therefore, ordinary dynamic scaling and a fortiori, dynamic multifractality, are complicated by the presence of sweeping terms. For example, in  $S_p(r=0, \tau)$  (obtained from measurements of velocity differences at a given point at finite values of the time difference) the kinematic exponent  $z = 1$  arising from sweeping occurs.[10] However, in the (Galilean

invariant) convective derivative which occurs in the equation above in the  $\tau \rightarrow 0$  limit only the intrinsic, dynamical exponent occurs. Thus the exact equation neatly picks out the intrinsic dynamical exponent. We will consider the implications of the two intermittency exponents after the corresponding derivation for the three-dimensional problem.

**Derivation for the Navier-Stokes equation:** We consider the Navier-Stokes equation for the velocity field  $\vec{u}(\vec{x}, t)$  driven by a stochastic driving force  $\vec{f}(\vec{x}, t)$  with zero mean and variance given by

$$\langle \hat{f}_i(\vec{k}) \hat{f}_j(\vec{k}') \rangle = P_{ij}(\vec{k}) D(\vec{k}) \delta_{\vec{k}+\vec{k}', \vec{0}} \delta(t-t') \quad (11)$$

where  $P_{ij}(\vec{k})$  is the transverse projection operator given by  $\delta_{ij} - (k_i k_j / k^2)$ . The noise covariance  $D(\vec{k})$  is assumed to be peaked around  $k_0 \sim 1/L$  with a narrow width. In contrast to the Burgers equation the detailed form of the noise correlation is not crucial in the 3d problem; the noise maintains a fully-developed turbulent state and allows one to define averages as noise ensemble averages. We will find it useful to define the quantity

$$\epsilon_{ij} \equiv \nu \partial_\ell u_i \partial_\ell u_j . \quad (12)$$

The dissipation rate  $\epsilon$  (cf. Eq. (1)) of an incompressible fluid obeys the relation

$$\epsilon = \epsilon_{ii} - \nu \nabla^2 \tilde{p} \quad (13)$$

where  $\tilde{p} = p/\rho$  ( $\rho$  is the constant density) and the summation convention of summing over repeated indices is used. We remark that  $\langle \epsilon_{ij} \rangle \propto \delta_{ij}$  in isotropic turbulence.

We use the notation  $\vec{x} = \vec{R} + (1/2)\vec{r}$ ,  $t = T + (1/2)\tau$  and  $\vec{x}' = \vec{R} - (1/2)\vec{r}$ ,  $t' = T - (1/2)\tau$ . Multiplying the Navier-Stokes equation for  $u_i = u_i(\vec{x}, t)$  by  $u_i$  and summing over  $i$  and using  $\nu u_i \nabla^2 u_i = \nu \nabla^2 (u^2/2) - \epsilon_{ii}$  one finds

$$-\epsilon_{ii} + \nu \nabla^2 (u^2/2) + \vec{f} \cdot \vec{u} = u_i \partial_t u_i + u_i u_l \partial_l u_i + u_i \partial_i \tilde{p} . \quad (14)$$

Again we write a similar equation for  $u'_i = u_i(\vec{x}', t')$  and multiply the two equations and average over the homogenous, steady state of isotropic turbulence leading to

$$\langle \epsilon_{ii} \epsilon'_{jj} \rangle - \frac{\nu}{2} \nabla_r^2 \langle \epsilon_{ii} u'^2 + \epsilon'_{ii} u^2 \rangle + \frac{\nu^2}{4} \nabla_r^2 \nabla_r^2 \langle u^2 u'^2 \rangle + \text{noise terms}$$

$$\begin{aligned}
&= -\frac{1}{4} \frac{\partial^2}{\partial \tau^2} \langle u^2 u'^2 \rangle - \frac{1}{2} \frac{\partial^2}{\partial \tau \partial r_i} \langle (u_i + u'_i) u^2 u'^2 \rangle - \frac{1}{4} \frac{\partial^2}{\partial r_i \partial r_j} \langle u_i u_j u^2 u'^2 \rangle \\
&\quad + \text{pressure terms} .
\end{aligned} \tag{15}$$

The second and third terms on the left-hand side are negligible in the inertial range and will be suppressed hereafter; the velocity terms clearly have the same form as in the Burgers equation consisting of a second derivative with respect to  $r$  of the sixth-order structure function and a second derivative with respect to  $\tau$  of a fourth-order structure function. We have not displayed the noise and pressure terms since this calculation merely illustrates the simple steps involved in the derivation. We will discuss them in the Galilean-invariant form given next.

The manifestly Galilean-invariant form is somewhat more difficult to obtain in the  $3-d$  problem. We proceed from the Navier-Stokes equation in two different ways and add the results appropriately. First multiply the Navier-Stokes equation for  $u_i$  and  $u'_i$  by  $\delta u_i \equiv u_i(\vec{x}, t) - \vec{u}_i(\vec{x}', t')$  and multiply the two equations to obtain

$$\delta u_i (\nu \nabla^2 u_i + f_i) \delta u_j (\nu \nabla'^2 u'_j + f'_j) = \delta u_i \left( \frac{Du_i}{Dt} + \partial_i \tilde{p} \right) \delta u_j \left( \frac{Du'_j}{Dt'} + \partial'_j \tilde{p}' \right) . \tag{16}$$

We now average this equation over the isotropic, homogeneous, steady state of turbulence. On the left-hand side we obtain, in addition to the noise terms,  $-\langle \epsilon_{ii} \epsilon'_{jj} + \epsilon_{ij} \epsilon'_{ij} \rangle$  as the only terms which survive in the inertial range. We do a similar set of manipulations with the Navier-Stokes equation for  $u_i$  multiplied by  $\delta u_j$  and obtain

$$\delta u_j (\nu \nabla^2 u_i + f_i) \delta u_j (\nu \nabla'^2 u'_i + f'_i) = \delta u_j \left( \frac{Du_i}{Dt} + \partial_i \tilde{p} \right) \delta u_j \left( \frac{Du'_i}{Dt'} + \partial'_i \tilde{p}' \right) . \tag{17}$$

We add Eqn. (16) to twice Eqn. (17) and average the resulting equation over the turbulent state. The right-hand side of the resultant equation includes, apart from the pressure terms,

$$\langle \delta u_i \frac{Du_i}{Dt} \delta u_j \frac{Du'_j}{Dt'} \rangle + 2 \langle \delta u_j \frac{Du_i}{Dt} \delta u_j \frac{Du'_i}{Dt'} \rangle .$$

We can show after some algebraic manipulations involving kinematic relations which are the three-dimensional generalizations of those displayed in

Equations (8) and (9) that these terms correspond to

$$(1/16)(\partial^2/\partial r_i \partial r_j) \langle \delta u_i \delta u_j (\delta \vec{u} \cdot \delta \vec{u})^2 \rangle - (1/4)(D^2/D\tau^2) \langle (\delta \vec{u} \cdot \delta \vec{u})^2 \rangle$$

where  $D^2/D\tau^2 \langle f \rangle = \partial^2/\partial \tau^2 \langle f \rangle + \partial^2/\partial \tau \partial r_i \langle (u_i + u'_i) f \rangle + (1/4) \partial^2/\partial r_i \partial r_j \langle (u_i + u'_i)(u_j + u'_j) f \rangle$ . This shows the occurrence of the two key structure function derivatives. We have evaluated the pressure terms and find

$$\begin{aligned} 2\langle \epsilon_{ii} \epsilon'_{jj} + 2\epsilon_{ij} \epsilon'_{ij} \rangle + \text{noise terms} &= \frac{1}{16} \frac{\partial^2}{\partial r_i \partial r_j} \langle \delta u_i \delta u_j (\delta \vec{u} \cdot \delta \vec{u})^2 \rangle - \frac{1}{4} \frac{D^2}{D\tau^2} \langle (\delta \vec{u} \cdot \delta \vec{u})^2 \rangle \\ &+ \langle 2\delta u_i \delta u_j \partial_i \tilde{p} \partial'_j \tilde{p}' + \delta u_i \delta u_i \partial_j \tilde{p} \partial'_j \tilde{p}' - \frac{D}{D\tau} \langle (\partial_i \tilde{p} + \partial'_i \tilde{p}') \delta u_i \delta \vec{u} \cdot \delta \vec{u} \rangle \\ &+ \frac{1}{2} \partial_{r_\ell} \langle \delta u_\ell \delta \vec{u} \cdot \delta \vec{u} \delta u_i (\partial_i \tilde{p} - \partial'_i \tilde{p}') \rangle . \end{aligned} \quad (18)$$

We emphasize that apart from terms negligible in the limit  $\nu \rightarrow 0$  the above equation is exact (the neglected terms have been evaluated.) To complete our discussion we must consider the noise and pressure terms in Eq. (18). We first note that

$$\langle \epsilon_{ii} \epsilon'_{jj} \rangle = \langle \epsilon \epsilon' \rangle + \nu \nabla_r^2 \langle \epsilon \tilde{p}' + \epsilon' \tilde{p} \rangle + \nu^2 \nabla_r^2 \nabla_r^2 \langle \tilde{p} \tilde{p}' \rangle$$

which is equal to  $\langle \epsilon \epsilon' \rangle$  in the inertial range. We expect the diagonal elements of  $\langle \partial_i u_\ell \partial_j u_\ell \rangle$  to yield the most singular terms and these survive in the  $\nu \rightarrow 0$  limit which cuts off the short-distance singularities. With this observation we see that only the terms with  $i = j$  contribute in  $\langle \epsilon_{ij} \epsilon'_{ij} \rangle$ ; using the isotropy of the turbulent state we find that the left-hand side of Eq.(18) reduces to  $(10/3) \langle \epsilon \epsilon' \rangle$  in the inertial range. The noise terms can be evaluated and the non-trivial term in the  $\tau \rightarrow 0^+$  limit is precisely  $(10/3) \langle \epsilon \rangle^2$ . Thus the left-hand side of Eq. (18) is  $(10/3) [\langle \epsilon \epsilon' \rangle - \langle \epsilon \rangle^2]$  in the inertial range. Finally, we observe that the pressure terms depend upon derivatives of the pressure  $\partial_i p$ . The kernel in the inversion of  $\nabla^2 p = -\partial_i u_j \partial_j u_i$  is Coulombic and long-ranged; however, the pressure-derivative terms can be written with a kernel which is dipolar and hence, shorter-ranged leading to more convergent integrals. Thus writing the pressure contributions in terms of velocities one concludes plausibly that these contributions will not be more dominant than those due to the terms  $D^2 S_4 / D\tau^2$  and  $\partial^2 S_6 / \partial r^2$ .

We discuss our results next. The behavior of  $\langle \epsilon \epsilon' \rangle$  is thus determined by the two terms in the limit  $\tau = 0$ , analogous to those in the Burgers equation.



The term  $\partial_{r_i}\partial_{r_j}\langle\delta u_i\delta u_j(\delta\vec{u}\cdot\delta\vec{u})^2\rangle$  yields  $\mu_1 = 2 - \zeta_6$ . The dynamical term,  $D^2/D\tau^2\langle\delta\vec{u}\cdot\delta\vec{u}\rangle^2$ , yields the identification  $\mu_2 = z_4'' - \zeta_4$ . This provides a transparent derivation directly from the Navier-Stokes equation of the two dominant exponents characterizing dissipation-rate correlations, one which depends purely on the static structure function exponent  $(2 - \zeta_6)$  and the other which involves dynamical behavior  $(z_4'' - \zeta_4)$ .

We proceed further, motivated by a similar strategy in the theory of phase transitions[12], and make an *ansatz* that the two terms are equally dominant. This leads to the identification[13]

$$2 - \zeta_6 = z_4'' - \zeta_4 \quad . \quad (19)$$

This relation connects multifractality in spatial correlations to multifractality in temporal correlations.[9] We then obtain the Kolmogorov result for the leading dissipation rate correlation exponent,  $2 - \zeta_6$ .

An earlier paper by us[11] provided a justification for the two exponents for the dissipation rate correlations using detailed equations for various structure functions explicitly, identifying terms in different equations and invoking plausible comparisons with the Burgers equation result. We note that other relations such as  $\mu = 2\zeta_2 - \zeta_4$  have been proposed in the literature.[14] Other derivations of the Kolmogorov relation include one based on fusion rules for equal time multipoint correlation functions by L'vov and Procaccia[15] who also pointed out another scenario which yields  $\mu = 2\zeta_2 - \zeta_4$  as in ref. [14].

We point out that in our equation the subtracted correlation,  $\langle\epsilon\epsilon'\rangle - \langle\epsilon\rangle^2$ , arises naturally. In the inertial range we expect the term  $(L/r)^\mu$  to dominate over the constant  $\langle\epsilon\rangle^2$  term. However, this might require  $r \ll L$  since  $\mu$  is small and this renders the experimental determination of  $\mu$  more difficult. However, for  $r \ll L$  whether one subtracts or not one should obtain the correct exponent as expected from these theoretical considerations, and this is clear in Ref. [4].

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$$\langle \hat{f}(k, t) \hat{\mathcal{F}}[\hat{u}(q, t)] \rangle = \frac{1}{2} \hat{D}(k) \langle \delta \hat{\mathcal{F}} / \delta \hat{u}(-k, t) \rangle$$

where  $D(k)$  is the variance of the noise.

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- [13] This ansatz relating dynamical and static exponents is supported by an exact equation for  $S_4$  in the stochastic Burgers problem which we have derived:

$$\frac{D^2 S_4}{D\tau^2} = \frac{1}{20} \frac{\partial^2 S_6}{\partial r^2} - 6 \frac{D}{D\tau} \langle \delta\epsilon(\delta u)^2 \rangle + \dots \quad .$$

This makes the identification of  $\zeta_6 - 2$  with  $\zeta_4 - z_2''$  plausible with both exponents being determined by the correlation of the dissipation rate with velocity differences. A similar equation can also be derived for the Navier-Stokes problem.

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